Ideal Fermi gas

Assume, the particles have a quadratic dispersion $\varepsilon_k = \frac{k^2}{2m}$

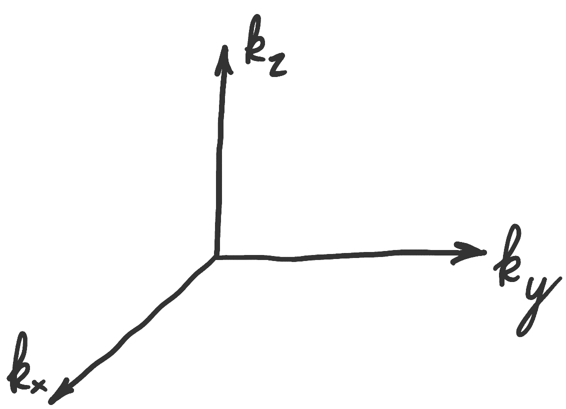
Let's first find the Fermi energy and Fermi momentum at $T=0$

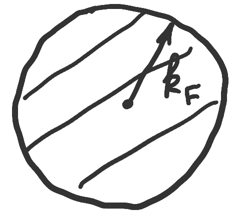
$$f = \begin{cases} 1, & \varepsilon < \varepsilon_F \\ 0, & \varepsilon > \varepsilon_F \end{cases}$$

One may say for momentum

$$f = \begin{cases} 1, & k < k_F \\ 0, & k > k_F \end{cases}$$

The total number of states in a system of volume V:





$g = 2S+1$
– spin degeneracy

$$g\,\frac{4}{3}\pi k_F^3 V \frac{1}{(2\pi\hbar)^3} = N \to \frac{g\,k_F^3}{6\pi^2\hbar^3} = \frac{N}{V}$$

$$\to k_F = \left(\frac{6\pi^2\hbar^3 n}{g}\right)^{\frac{1}{3}}$$

$$\varepsilon_F = \frac{k_F^2}{2m} = \frac{(6\pi^2\hbar^3 n)^{\frac{2}{3}}}{2m\,g^{\frac{2}{3}}}$$

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If $T \ll \varepsilon_F$ $(= T_F)$, this Fermi gas is called degenerate (Almost always in solids)

If $T \gg \varepsilon_F$, then it behaves like a classical ideal gas, with occupation numbers

$$n_k \propto e^{-\frac{k^2}{2mT}}$$

Consider now a Fermi gas at an arbitrary temperature T

$g \dfrac{d\vec{p}\, dV}{(2\pi\hbar)^3}$ — the number of states corresponding to the element of momentum space $d\vec{p}$ and volume dV

The DoS in 3D:

$$V \frac{g \cdot 4\pi p^2 dp}{(2\pi\hbar)^3} = \nu\, d\varepsilon \rightarrow \nu = \frac{4\pi g p^2}{(2\pi\hbar)^3 v} V \quad \left(v = \frac{d\varepsilon}{dp}\right)$$

Use $p^2 = 2m\varepsilon$, $v = \sqrt{\frac{2\varepsilon}{m}}$

$$\nu = \frac{g V m^{\frac{3}{2}}}{2^{\frac{1}{2}} \pi^2 \hbar^3}$$

Total number of particles

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$$N = \frac{g V m^{\frac{3}{2}}}{2^{\frac{1}{2}} \pi^2 \hbar^3} \int_0^\infty \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\frac{\varepsilon-\mu}{T}} + 1}$$

– this is the equation from which we may find the chemical potential μ as a function of μ and T.

Consider the integral

// Note: $\beta = \frac{1}{T}$ is a common notation in the literature

$$\int_0^\infty \frac{\varphi(\varepsilon)\, d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1} = T \int_{-\frac{\mu}{T}}^{\infty} \frac{\varphi(\mu T + x)}{e^x + 1} = \left(\begin{array}{l} x = \beta(\varepsilon - \mu) \\ \varepsilon = \mu + xT \end{array} \right)$$

$$= \underbrace{T \int_{-\frac{\mu}{T}}^{0} \frac{\varphi(\mu + Tx)}{e^x + 1} dx}_{\mathrm{I}} + \underbrace{T \int_0^\infty \frac{\varphi(\mu + Tx)}{e^x + 1} dx}_{\mathrm{II}}$$

$$\mathrm{I} = T \int_{-\frac{\mu}{T}}^{0} \frac{\varphi(\mu + Tx)}{e^x + 1} dx = T \int_0^{\frac{\mu}{T}} \frac{\varphi(\mu - Tx)}{e^{-x} + 1} dx =$$

$$\frac{1}{e^{-x}+1} = 1 - \frac{1}{e^x + 1}$$

$$= \int_0^\mu \varphi(z)\, dz - \int_0^{\frac{\mu}{T}} \frac{\varphi(\mu - Tx)}{e^x + 1} dx$$

$(z = \mu - Tx)$

Combining I and II,

$$\int_0^\infty \frac{\varphi(\varepsilon)\,d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \int_0^\mu \varphi(z)\,dz + T\int_0^\infty \frac{\varphi(\mu+Tx)}{e^x+1}\,dx - T\int_0^{\frac{\mu}{T}} \frac{\varphi(\mu-Tx)}{e^x+1}\,dx$$

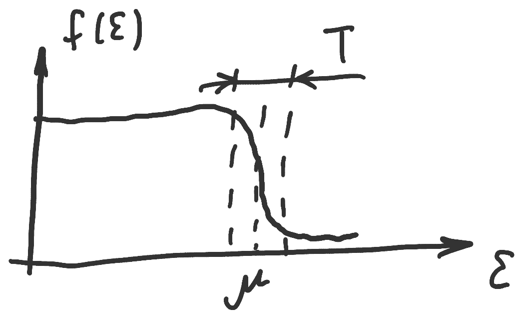
Now use a small parameter $\frac{T}{\mu} \ll 1$
That allows one to extend the upper limit of integration in the second integral, $\frac{\mu}{T} \to \infty$

$$\varphi(\mu+Tx) \approx \varphi(\mu) + Tx\,\varphi'(\mu) + \dots$$

$$\varphi(\mu+Tx) - \varphi(\mu-Tx) \approx 2Tx\,\varphi'(\mu) + \dots$$

Then

$$\int_0^\infty \frac{\varphi(\varepsilon)\,d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \int_0^\mu \varphi(z)\,dz + 2T^2\varphi'(\mu)\underbrace{\int_0^\infty \frac{x\,dx}{e^x+1}}_{=\frac{\pi^2}{12}} + \dots$$



$$\boxed{\int_0^\infty \frac{\varphi(\varepsilon)\,d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \int_0^\mu \varphi(\varepsilon)\,d\varepsilon + \frac{\pi^2 T^2}{6}\varphi'(\mu) + \dots}$$

– Sommerfeld expansion

apply this to $\varphi(\varepsilon) = \sqrt{\varepsilon}$

$$\int_0^{\mu} \sqrt{\varepsilon}\, d\varepsilon = \frac{2}{3}\mu^{\frac{3}{2}}$$

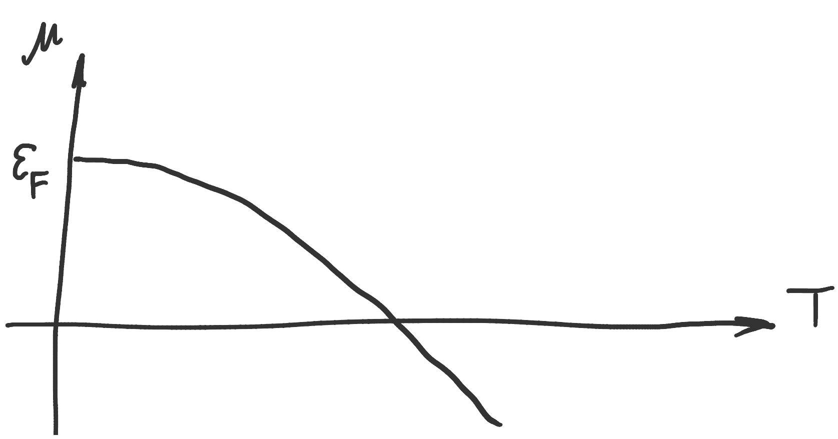
$$\varphi'(\mu) = \frac{1}{2\mu^{\frac{1}{2}}}$$

$$\frac{N}{V} = \frac{g V m^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3}\left(\frac{2}{3}\mu^{\frac{3}{2}} + \frac{\pi^2}{12}\mu^{\frac{3}{2}}\left(\frac{T}{\mu}\right)^2 + \dots\right)$$

$$\underbrace{\frac{3}{2}\frac{N}{V}\frac{2^{\frac{1}{2}}\pi^2\hbar^3}{g V m^{\frac{3}{2}}}}_{\text{That has to be } \varepsilon_F^{\frac{3}{2}}} = \mu^{\frac{3}{2}}\left(1 + \frac{\pi^2}{8}\left(\frac{T}{\mu}\right)^2 + \dots\right)$$

$$\mu = \varepsilon_F\left(1 - \frac{2}{3}\cdot\frac{\pi^2}{8}\left(\frac{T}{\varepsilon_F}\right)^2 + \dots\right)$$

$$\boxed{\mu \approx \varepsilon_F\left(1 - \frac{\pi^2}{6}\left(\frac{T}{\varepsilon_F}\right)^2 + \dots\right)}$$



Heat capacity of a metal

Energy $E = \int \nu(\varepsilon) f(\varepsilon) d\varepsilon =$

$$= \frac{g V m^{\frac{3}{2}}}{2^{\frac{1}{2}} \pi^2 \hbar^3} \int \frac{\varepsilon^{\frac{3}{2}} d\varepsilon}{e^{\frac{\varepsilon-\mu}{T}} + 1}$$

Apply Sommerfeld expansion with $\varphi(\varepsilon) = \varepsilon^{\frac{3}{2}}$

We need the second term with $\frac{\pi^2 T^2}{6} \varphi'(\mu)$

$$E = \underbrace{\frac{g V m^{\frac{3}{2}}}{2^{\frac{1}{2}} \pi^2 \hbar^3}}_{\frac{3}{2} \frac{N}{V} \varepsilon_F^{-\frac{3}{2}}} \left(\text{const} + \frac{\pi^2}{4} T^2 \mu^{\frac{1}{2}} + \dots \right)$$

$$\boxed{C \approx \frac{dE}{dT} = \frac{3\pi^2}{4} \frac{N}{V} \frac{T}{\varepsilon_F}}$$

Note: $C \propto T$ and this has a simple qualitative interpretation

Equations of various processes

Reminder: $\Omega_k = -T \ln\left(1 + e^{\frac{\mu - \varepsilon_k}{T}}\right)$

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Sum („integrate") wrt all states:

$$\Omega = -\frac{VgTm^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3} \underbrace{\int_0^\infty \sqrt{\varepsilon}\,\ln\left(1+e^{\frac{\mu-\varepsilon}{T}}\right) d\varepsilon}_{I}$$

$$I = \underbrace{-\frac{2}{3}\varepsilon^{\frac{3}{2}}\ln\left(1+e^{\frac{\mu-\varepsilon}{T}}\right)\Bigg|_{\varepsilon=0}^{\varepsilon=\infty}}_{\text{Vanishes}} + \frac{2}{3}\int_0^\infty \frac{\varepsilon^{\frac{3}{2}}d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$\Omega = -\frac{2}{3}\frac{VgTm^{\frac{3}{2}}}{2^{\frac{1}{2}}\pi^2\hbar^3}\int_0^\infty \frac{\varepsilon^{\frac{3}{2}}d\varepsilon}{e^{\frac{\varepsilon-\mu}{T}}+1} = -\frac{2}{3}E$$

Since $\Omega = -PV$, $\boxed{PV = \frac{2}{3}E}$

Note: this is exact!!!

// Reminder: in a classical ideal gas $E = \frac{3}{2}NT$

$$\Omega = VT^{\frac{5}{2}} f\left(\frac{\mu}{T}\right) \qquad \longrightarrow$$

$$\begin{cases} S = -\left(\frac{\partial\Omega}{\partial T}\right)_{V,\mu} = -\frac{5}{2}VT^{\frac{3}{2}}f\left(\frac{\mu}{T}\right) + VT^{\frac{5}{2}}\frac{\mu}{T^2}f'\left(\frac{\mu}{T}\right) & (1) \\ N = -\left(\frac{\partial\Omega}{\partial\mu}\right)_{T,V} = -VT^{\frac{3}{2}}f'\left(\frac{\mu}{T}\right) & (2) \end{cases}$$

$$\frac{S}{N} = \omega\left(\frac{\mu}{T}\right)$$

Thus, in an adiabatic process $\frac{\mu}{T} = \text{const}$

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Then from Eq. (1) $\quad VT^{\frac{3}{2}} = \text{const}$

Using that $\Omega = -PV = VT^{\frac{5}{2}} f\left(\frac{\mu}{T}\right)$

$$\boxed{PV^{\frac{5}{3}} = \text{const}}$$

The equation of adiabat coincides with that for a monoatomic gas ; $PV^{\gamma} = \text{const}$ where $\gamma = \frac{C_p}{C_v}$ $\quad$ ($= \frac{5}{3}$ for a monoatomic gas)